ベクトル解析の公式

1. 記法と定義

Einstein の縮約記法： $a_i b_i \equiv \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ (同じ添字については和をとる)

Kronecker のデルタ： $\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$

Eddington のイプシロン： $\epsilon_{ijk} = \begin{cases} +1 & \big((i,j,k) = (1,2,3),(2,3,1),(3,1,2)\big) \\ -1 & \big((i,j,k) = (1,3,2),(2,1,3),(3,2,1)\big) \\ 0 & (\text{otherwise}) \end{cases}$

2. 基本的な計算

(1) $\delta_{ij} a_j = a_i$

(2) $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}$

(3) $\epsilon_{ijk} a_j a_k = 0$

(4) $\epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

(5) $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij} a_i b_j = a_i b_i = \boldsymbol{b} \cdot \boldsymbol{a}$ (内積)

(6) $(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k = -\epsilon_{ikj} b_k a_j = -(\boldsymbol{b} \times \boldsymbol{a})_i$ (外積)

3. ベクトル解析の公式

3.1 ベクトルの公式

(1) $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \det\,[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$ (スカラー三重積)

(2) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$ (ベクトル三重積)

(3) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) = \boldsymbol{a} \cdot \big(\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})\big)$ (スカラー四重積)

(4) $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \big(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{d})\big)\,\boldsymbol{c} - \big(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\big)\,\boldsymbol{d} = \big(\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{d})\big)\,\boldsymbol{b} - \big(\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{d})\big)\,\boldsymbol{a}$ (ベクトル四重積)

(5) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$ (Jacobi 恒等式)

3.2 ベクトル場の微分公式

(1) $\mathrm{grad}\, r = \nabla r = \dfrac{\boldsymbol{r}}{r}$

(2) $\mathrm{grad}\, f(r) = \nabla f(r) = f'(r)\dfrac{\boldsymbol{r}}{r}$

(3) $\mathrm{div}(f\boldsymbol{v}) = \nabla \cdot (f\boldsymbol{v}) = (\nabla f) \cdot \boldsymbol{v} + f(\nabla \cdot \boldsymbol{v})$

(4) $\mathrm{rot}(f\boldsymbol{v}) = \nabla \times (f\boldsymbol{v}) = (\nabla f) \times \boldsymbol{v} + f(\nabla \times \boldsymbol{v})$

(5) $\mathrm{grad}(\boldsymbol{v} \cdot \boldsymbol{w}) = \nabla(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{w}) + \boldsymbol{w} \times (\nabla \times \boldsymbol{v})$

(6) $\mathrm{div}(\boldsymbol{v} \times \boldsymbol{w}) = \nabla \cdot (\boldsymbol{v} \times \boldsymbol{w}) = (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{w} - \boldsymbol{v} \cdot (\nabla \times \boldsymbol{w})$

(7) $\mathrm{rot}(\boldsymbol{v} \times \boldsymbol{w}) = \nabla \times (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v}(\nabla \cdot \boldsymbol{w}) - \boldsymbol{w}(\nabla \cdot \boldsymbol{v}) + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{w}$

(8) $\mathrm{rot}(\mathrm{grad}\, f) = \nabla \times \nabla f = \boldsymbol{0}$ (勾配の回転はゼロ)

(9) $\mathrm{div}(\mathrm{rot}\ \boldsymbol{v}) = \nabla \cdot (\nabla \times \boldsymbol{v}) = 0$ (回転の発散はゼロ)

(10) $\mathrm{rot}(\mathrm{rot}\ \boldsymbol{v}) = \nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$

3.3 ベクトル場の積分公式

(1) $\displaystyle\int_V (\nabla \cdot \boldsymbol{v})\, dV = \oint_{S=\partial V} \boldsymbol{v} \cdot d\boldsymbol{S}$ (Gauss の発散定理)

(2) $\displaystyle\int_S (\nabla \times \boldsymbol{v}) \cdot d\boldsymbol{S} = \oint_{C=\partial S} \boldsymbol{v} \cdot d\boldsymbol{s}$ (Stokes の定理)

ベクトル解析の公式の証明

3.1 ベクトルの公式

(1) スカラー三重積 $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})=\det\left[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\right]$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=a_i\,(\boldsymbol{b}\times\boldsymbol{c})_i=a_i\,\epsilon_{ijk}b_jc_k=\epsilon_{ijk}a_ib_jc_k=\det\left[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\right]\ .$$

(2) ベクトル三重積 $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$

$$\begin{aligned}\big(\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\big)_i=\epsilon_{ijk}a_j\ (\boldsymbol{b}\times\boldsymbol{c})_k&=\epsilon_{ijk}a_j\,(\epsilon_{kmn}b_mc_n)\\&=\epsilon_{ijk}\epsilon_{kmn}\,a_jb_mc_n\\&=\epsilon_{kij}\epsilon_{kmn}\,a_jb_mc_n\\&=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\,a_jb_mc_n\\&=a_jb_ic_j-a_jb_jc_i\\&=(\boldsymbol{a}\cdot\boldsymbol{c})\,b_i-(\boldsymbol{a}\cdot\boldsymbol{b})\,c_i\\&=\big((\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}\big)_i\ .\end{aligned}$$

(3) スカラー四重積 $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})=\boldsymbol{a}\cdot\big(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})\big)$

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})&=(\epsilon_{ijk}a_jb_k)(\epsilon_{imn}c_md_n)=\epsilon_{ijk}\epsilon_{imn}a_jb_kc_md_n=(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})a_jb_kc_md_n\\&=a_jc_jb_kd_k-a_jd_jb_kc_k=(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})\\&=a_j(c_jb_kd_k-d_jb_kc_k)=\boldsymbol{a}\cdot\big((\boldsymbol{b}\cdot\boldsymbol{d})\,\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{d}\big)=\boldsymbol{a}\cdot\big(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})\big)\ .\end{aligned}$$

特に，$\boldsymbol{c}=\boldsymbol{a}$,$\boldsymbol{d}=\boldsymbol{b}$ とおくと，Lagrange の恒等式を得る．

$$|\boldsymbol{a}\times\boldsymbol{b}|^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2\ .$$

(4) ベクトル四重積 $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=\big(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{d})\big)\,\boldsymbol{c}-\big(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\big)\,\boldsymbol{d}=\big(\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{d})\big)\,\boldsymbol{b}-\big(\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})\big)\,\boldsymbol{a}$

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})&=\big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}\big)\,\boldsymbol{c}-\big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\big)\,\boldsymbol{d}\\&=\big(\boldsymbol{d}\cdot(\boldsymbol{a}\times\boldsymbol{b})\big)\,\boldsymbol{c}-\big(\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})\big)\,\boldsymbol{d}\\&=\big(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{d})\big)\,\boldsymbol{c}-\big(\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\big)\,\boldsymbol{d}\ .\\(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})&=-(\boldsymbol{c}\times\boldsymbol{d})\times(\boldsymbol{a}\times\boldsymbol{b})\\&=-\big((\boldsymbol{c}\times\boldsymbol{d})\cdot\boldsymbol{b}\big)\,\boldsymbol{a}+\big((\boldsymbol{c}\times\boldsymbol{d})\cdot\boldsymbol{a}\big)\,\boldsymbol{b}\\&=-\big(\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})\big)\,\boldsymbol{a}+\big(\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{d})\big)\,\boldsymbol{b}\ .\end{aligned}$$

(5) Jacobi 恒等式 $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{0}$

$$\begin{aligned}&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})\\=&\big((\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}\big)+\big((\boldsymbol{b}\cdot\boldsymbol{a})\,\boldsymbol{c}-(\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}\big)+\big((\boldsymbol{c}\cdot\boldsymbol{b})\,\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{a})\,\boldsymbol{b}\big)\\=&\,\boldsymbol{0}\ .\end{aligned}$$

3.2 ベクトル場の微分公式

(1) $\mathrm{grad}\, r = \nabla r = \dfrac{\boldsymbol{r}}{r}$

$$(\nabla r)_i = \partial_i r = \partial_i \sqrt{r_j r_j} = \frac{\partial_i (r_j r_j)}{2\sqrt{r_j r_j}} = \frac{2\delta_{ij} r_j}{2\sqrt{r_j r_j}} = \frac{r_i}{r} = \left(\frac{\boldsymbol{r}}{r}\right)_i \ .$$

(2) $\mathrm{grad}\, f(r) = \nabla f(r) = f'(r)\dfrac{\boldsymbol{r}}{r}$

$$\big(\nabla f(r)\big)_i = \partial_i f(r) = \frac{\partial r}{\partial r_i}\frac{d}{dr} f(r) = f'(r)\frac{r_i}{r} = \Big(f'(r)\frac{\boldsymbol{r}}{r}\Big)_i \ .$$

(3) $\mathrm{div}(f\boldsymbol{v}) = \nabla \cdot (f\boldsymbol{v}) = (\nabla f)\cdot \boldsymbol{v} + f(\nabla \cdot \boldsymbol{v})$

$$\nabla \cdot (f\boldsymbol{v}) = \partial_i (f\boldsymbol{v})_i = \partial_i (f v_i) = (\partial_i f) v_i + f(\partial_i v_i) = (\nabla f)_i v_i + f(\partial_i v_i) = (\nabla f)\cdot \boldsymbol{v} + f(\nabla \cdot \boldsymbol{v}) \ .$$

(4) $\mathrm{rot}(f\boldsymbol{v}) = \nabla \times (f\boldsymbol{v}) = (\nabla f) \times \boldsymbol{v} + f(\nabla \times \boldsymbol{v})$

$$\begin{aligned}\big(\nabla \times (f\boldsymbol{v})\big)_i &= \epsilon_{ijk}\partial_j (f\boldsymbol{v})_k = \epsilon_{ijk}\partial_j (f v_k) = \epsilon_{ijk}(\partial_j f) v_k + \epsilon_{ijk} f(\partial_j v_k) \\ &= \epsilon_{ijk}(\nabla f)_j v_k + f\epsilon_{ijk}\partial_j v_k = \big((\nabla f)\times \boldsymbol{v}\big)_i + f(\nabla \times \boldsymbol{v})_i = \big((\nabla f)\times \boldsymbol{v} + f(\nabla \times \boldsymbol{v})\big)_i \ .\end{aligned}$$

(5) $\mathrm{grad}(\boldsymbol{v}\cdot \boldsymbol{w}) = \nabla(\boldsymbol{v}\cdot \boldsymbol{w}) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + \boldsymbol{v}\times(\nabla\times\boldsymbol{w}) + \boldsymbol{w}\times(\nabla\times\boldsymbol{v})$

$$\begin{aligned}&\big(\boldsymbol{v}\times(\nabla\times\boldsymbol{w}) + \boldsymbol{w}\times(\nabla\times\boldsymbol{v})\big)_i = \epsilon_{ijk}\big(v_j(\nabla\times\boldsymbol{w})_k + w_j(\nabla\times\boldsymbol{v})_k\big) \\ =\ & \epsilon_{kij}\epsilon_{kmn}(v_j\partial_m w_n + w_j\partial_m v_n) = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(v_j\partial_m w_n + w_j\partial_m v_n) \\ =\ & v_j\partial_i w_j - v_j\partial_j w_i + w_j\partial_i v_j - w_j\partial_j v_i\end{aligned}$$

より,

$$\begin{aligned}&\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + \boldsymbol{v}\times(\nabla\times\boldsymbol{w}) + \boldsymbol{w}\times(\nabla\times\boldsymbol{v})\big)_i \\ =\ & v_j\partial_j w_i + w_j\partial_j v_i + v_j\partial_i w_j - v_j\partial_j w_i + w_j\partial_i v_j - w_j\partial_j v_i \\ =\ & v_j\partial_i w_j + w_j\partial_i v_j = \partial_i(v_j w_j) = \big(\nabla(\boldsymbol{v}\cdot\boldsymbol{w})\big)_i \ .\end{aligned}$$

(6) $\mathrm{div}(\boldsymbol{v}\times\boldsymbol{w}) = \nabla\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w} - \boldsymbol{v}\cdot(\nabla\times\boldsymbol{w})$

$$\begin{aligned}\nabla\cdot(\boldsymbol{v}\times\boldsymbol{w}) &= \partial_i(\epsilon_{ijk}v_j w_k) = (\epsilon_{kij}\partial_i v_j)w_k - v_j\epsilon_{jik}\partial_i w_k \\ &= (\nabla\times\boldsymbol{v})_k w_k - v_j(\nabla\times\boldsymbol{w})_j = (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w} - \boldsymbol{v}\cdot(\nabla\times\boldsymbol{w}) \ .\end{aligned}$$

(7) $\mathrm{rot}(\boldsymbol{v}\times\boldsymbol{w}) = \nabla\times(\boldsymbol{v}\times\boldsymbol{w}) = \boldsymbol{v}(\nabla\cdot\boldsymbol{w}) - \boldsymbol{w}(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$

$$\begin{aligned}\big(\mathrm{rot}(\boldsymbol{v}\times\boldsymbol{w})\big)_i &= \epsilon_{ijk}\partial_j(\boldsymbol{v}\times\boldsymbol{w})_k = \epsilon_{ijk}\partial_j(\epsilon_{kmn}v_m w_n) = \epsilon_{ijk}\epsilon_{kmn}(v_m\partial_j w_n + w_n\partial_j v_m) \\ &= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(v_m\partial_j w_n + w_n\partial_j v_m) = v_i\partial_j w_j - v_j\partial_j w_i + w_j\partial_j v_i - w_i\partial_j v_j \\ &= \big(\boldsymbol{v}(\nabla\cdot\boldsymbol{w}) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} - \boldsymbol{w}(\nabla\cdot\boldsymbol{v})\big)_i \ .\end{aligned}$$

(8) $\mathrm{rot}(\mathrm{grad}\, f) = \nabla\times\nabla f = \boldsymbol{0}$

$$\big(\nabla\times(\nabla f)\big)_i = \epsilon_{ijk}\partial_j(\nabla f)_k = \epsilon_{ijk}\partial_j\partial_k f = 0 \ .$$

(9) $\mathrm{div}(\mathrm{rot}\ \boldsymbol{v}) = \nabla\cdot(\nabla\times\boldsymbol{v}) = 0$

$$\nabla\cdot(\nabla\times\boldsymbol{v}) = \partial_i(\nabla\times\boldsymbol{v})_i = \partial_i\epsilon_{ijk}\partial_j v_k = \epsilon_{ijk}\partial_i\partial_j v_k = 0 \ .$$

(10) $\mathrm{rot}(\mathrm{rot}\ \boldsymbol{v}) = \nabla\times(\nabla\times\boldsymbol{v}) = \nabla(\nabla\cdot\boldsymbol{v}) - \nabla^2\boldsymbol{v}$

$$\begin{aligned}\big(\nabla\times(\nabla\times\boldsymbol{v})\big)_i &= \epsilon_{ijk}\partial_j(\nabla\times\boldsymbol{v})_k = \epsilon_{ijk}\partial_j(\epsilon_{kmn}\partial_m v_n) = \epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m v_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m v_n \\ &= \partial_j\partial_i v_j - \partial_j\partial_j v_i = \partial_i(\nabla\cdot\boldsymbol{v}) - \nabla^2 v_i = \big(\nabla(\nabla\cdot\boldsymbol{v}) - \nabla^2\boldsymbol{v}\big)_i \ .\end{aligned}$$

3.3 ベクトル場の積分公式

(1) Gauss の発散定理 $\displaystyle\int_V (\nabla\cdot\boldsymbol{v})\,dV = \oint_{S=\partial V} \boldsymbol{v}\cdot d\boldsymbol{S}$

領域 V を微小直方体 $V=(x,x+dx)\times(y,y+dy)\times(z,z+dz)$ にとると，

$$
\begin{aligned}
\int_V (\nabla\cdot\boldsymbol{v})\,dV &= (\nabla\cdot\boldsymbol{v})\,dxdydz \\
&= (\partial_x v_x + \partial_y v_y + \partial_z v_z)dxdydz \\
&= \left(\frac{v_x(x+dx,y,z)-v_x(x,y,z)}{dx} + \frac{v_y(x,y+dy,z)-v_y(x,y,z)}{dy} + \frac{v_z(x,y,z+dz)-v_z(x,y,z)}{dz}\right)dxdydz \\
&= v_x(x+dx,y,z)dydz - v_x(x,y,z)dydz + v_y(x,y+dy,z)dzdx - v_y(x,y,z)dzdx \\
&\qquad\qquad + v_z(x,y,z+dz)dxdy - v_z(x,y,z)dxdy \\
&= \boldsymbol{v}(x+dx,y,z)\cdot\boldsymbol{e}_x dydz + \boldsymbol{v}(x,y,z)\cdot(-\boldsymbol{e}_x dydz) + \boldsymbol{v}(x,y+dy,z)\cdot\boldsymbol{e}_y dzdx + \boldsymbol{v}(x,y,z)\cdot(-\boldsymbol{e}_y dzdx) \\
&\qquad\qquad + \boldsymbol{v}(x,y,z+dz)\cdot\boldsymbol{e}_z dxdy + \boldsymbol{v}(x,y,z)\cdot(-\boldsymbol{e}_z dxdy) \\
&= \sum_{S=\partial V} \boldsymbol{v}\cdot d\boldsymbol{S} = \oint_{S=\partial V} \boldsymbol{v}\cdot d\boldsymbol{S}\ .
\end{aligned}
$$

一般の領域 V については，V を微小直方体領域 V_i に分割して，各 V_i に上の結果を適用すればよい．

$$
\int_V (\nabla\cdot\boldsymbol{v})\,dV = \sum_i \int_{V_i} (\nabla\cdot\boldsymbol{v})\,dV = \sum_i \oint_{S_i=\partial V_i} \boldsymbol{v}\cdot d\boldsymbol{S} = \oint_{S=\partial V} \boldsymbol{v}\cdot d\boldsymbol{S}\ .
$$

最後の等号は，互いに接した微小領域では面の法線ベクトル ($d\boldsymbol{S}$) の向きが反対であるから表面積分が互いに打ち消し合い，V の表面 $S=\partial V$ での積分だけが残ることを用いた．

(2) Stokes の定理 $\displaystyle\int_S (\nabla\times\boldsymbol{v})\cdot d\boldsymbol{S} = \oint_{C=\partial S} \boldsymbol{v}\cdot d\boldsymbol{s}$

領域 (曲面) S を微小長方形 $S=(x,x+dx)\times(y,y+dy)$ にとり，$d\boldsymbol{S}=dS\,\boldsymbol{e}_z = dxdy\,\boldsymbol{e}_z$ とすると，

$$
\begin{aligned}
\int_S (\nabla\times\boldsymbol{v})\cdot d\boldsymbol{S} &= (\nabla\times\boldsymbol{v})\cdot\boldsymbol{e}_z\,dxdy \\
&= (\nabla\times\boldsymbol{v})_z\,dxdy \\
&= (\partial_x v_y - \partial_y v_x)\,dxdy \\
&= \left(\frac{v_y(x+dx,y,z)-v_y(x,y,z)}{dx} - \frac{v_x(x,y+dy,z)-v_x(x,y,z)}{dy}\right)dxdy \\
&= v_y(x+dx,y,z)dy - v_y(x,y,z)dy - v_x(x,y+dy,z)dx + v_x(x,y,z)dx \\
&= \boldsymbol{v}(x,y,z)\cdot\boldsymbol{e}_x dx + \boldsymbol{v}(x+dx,y,z)\cdot\boldsymbol{e}_y dy + \boldsymbol{v}(x,y+dy,z)\cdot(-\boldsymbol{e}_x dx) + \boldsymbol{v}(x,y,z)\cdot(-\boldsymbol{e}_y dy) \\
&= \sum_{C=\partial S} \boldsymbol{v}\cdot d\boldsymbol{s} = \oint_{C=\partial S} \boldsymbol{v}\cdot d\boldsymbol{s}\ .
\end{aligned}
$$

この結果は座標系の取り方に依らない幾何学的な結果であることに注意．一般の領域 S については，S を微小長方形領域 S_i に分割して，各 S_i に上の結果を適用すればよい．

$$
\int_S (\nabla\times\boldsymbol{v})\cdot d\boldsymbol{S} = \sum_i \int_{S_i} (\nabla\times\boldsymbol{v})\cdot d\boldsymbol{S} = \sum_i \oint_{C_i=\partial S_i} \boldsymbol{v}\cdot d\boldsymbol{s} = \oint_{C=\partial S} \boldsymbol{v}\cdot d\boldsymbol{s}\ .
$$

最後の等号は，互いに接した微小領域では線素ベクトル ($d\boldsymbol{s}$) の向きが反対であるから線積分が互いに打ち消し合い，S の境界線 $C=\partial S$ での積分だけが残ることを用いた．